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**A "KANE'S DYNAMICS" MODEL  
FOR THE ACTIVE RACK ISOLATION SYSTEM**

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Many microgravity space-science experiments require vibratory acceleration levels unachievable without active isolation. The Boeing Corporation's Active Rack Isolation System (ARIS) employs a novel combination of magnetic actuation and mechanical linkages, to address these isolation requirements on the International Space Station (ISS). ARIS provides isolation at the rack (International Standard Payload Rack, or ISPR) level.

Effective model-based vibration isolation requires (1) an appropriate isolation device, (2) an adequate dynamic (i.e., mathematical) model of that isolator, and (3) a suitable, corresponding controller. ARIS provides the ISS response to the first requirement. This paper presents one response to the second, in a state-space framework intended to facilitate an optimal-controls approach to the third. The authors use "Kane's Dynamics" to develop an state-space, analytical (algebraic) set of linearized equations of motion for ARIS.

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### **INTRODUCTION**

The vibratory acceleration levels currently achievable, without isolation on manned space structures, exceed those required by many space-science experiments (DeLombard, et al., 1997; NASA Specification Number SSP41000, Rev. D., 1995; DelBasso, 1996; Nelson, 1991). Various active isolation devices have been built to address this need. The first in space was called STABLE ("Suppression of Transient Accelerations By LEvitation"), which uses six independently-controlled Lorentz actuators to levitate and isolate at the experiment (or sub-experiment) level (Edberg, et al., 1996). It was successfully flight-tested on STS-73 (USML-02) in October 1995. Marshall Space Flight Center (MSFC) is developing a second-generation experiment-level isolation system (g-LIMIT: "GLovebox Integrated Microgravity Isolation Technology"), building on the technology developed for STABLE (Whorton, 1998). This compact system will isolate microgravity payloads in the Microgravity Science Glovebox (MSG).

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A second experiment-level isolation system, called MIM ("Microgravity Vibration Isolation Mount"), was launched in the Priroda laboratory module which docked with Mir in April 1996 (Hampton, et al., 1997). MIM uses eight Lorentz actuators, with centralized control. It has supported several materials science experiments since its implementation in May 1996. A modified version of MIM (MIM II) supported additional experiments on STS-85 in August 1997.

Boeing's Active Rack Isolation System (ARIS), in contrast to the above payload-isolation systems, has been designed to isolate at the rack level; an entire International Standard Payload Rack (ISPR) will be isolated by each copy of ARIS on ISS. The Risk-Mitigation Experiment (RME) for ARIS was conducted in September 1996 on STS-79 (Bushnell, 1996). Each of ARIS' eight electromechanical actuators requires a two rigid-body model; when the ISPR ("flotor") is included, the total isolation-system model contains 17 rigid bodies.

In order to provide effective model-based isolation, the task of controller design requires prior development of an adequate dynamic (i.e., mathematical) model of the isolation system. This paper presents a dynamic model of ARIS, in a state-space framework intended to facilitate design of an optimal controller. The chosen approach is the method of Thomas R. Kane ("Kane's method") (Kane and Levinson, 1995); the result is a state-space, analytical (algebraic) set of linearized equations of motion for ARIS.

### **THE CHOICE OF KANE'S METHOD**

There are fundamentally two avenues for deriving system dynamical equations of motion: vector methods and energy methods. Both avenues lead to scalar equations, but they have different starting points. Vector methods begin with vector equations proceeding from Newton's Laws of Motion; and energy methods, with scalar energy expressions. The former category uses approaches built around (1) Momentum Principles, (2) D'Alembert's Principle, or (3) Kane's Method; and the latter, (1) Hamilton's Canonical Equations, (2) the Boltzmann-Hamel Equations, (3) the Gibbs Equations, or (4) Lagrange's Equations.

Although some problems might lend themselves better to solution by other approaches, Kane's method appears in general to be distinctly advantageous for complex problems. As a rule, of the above approaches, those that lead to the simplest and most intuitive dynamical equations are the Gibbs Equations and Kane's Equations. And of those two approaches the latter is the more systematic and requires less labor. The reduction of labor is particularly evident when one seeks linearized equations of motion, as proved to be necessary in the present case (due to the otherwise excessive algebraic burden).

An overview of Kane's approach to developing linearized equations of motion is presented in (Hampton, et al., 1998), along with a summary of the relative advantages of the method. See Kane and Levinson (1979, 1985) for more extended treatments.

## DESCRIPTION OF ARIS

The total dynamical system  $\tilde{S}$  consists of the stator  $S$  (ISS and the integral frame, from the motion of which ARIS isolates the ISPR), the flotor  $F$  (the ISPR), eight electromechanical actuator assemblies, and the umbilicals. (See Figure 1.) The flotor is connected to the stator by the eight actuator assemblies, and by a variable number of umbilicals. The actuator assemblies also (and fundamentally) act as the vibration isolation devices.

Each actuator assembly consists of a Lorentz (voice-coil) actuator, an arm, an upper stinger, a push-rod, a lower stinger, and a position sensor. (See Fig. 2 for a kinematic diagram, and Fig. 3 for a CAD drawing, of a single actuator.) One end of each actuator arm is connected to the flotor through a cross-flexure which allows the flotor a single rotational degree of freedom with respect to the stator. The other end of the arm is connected to one end of the push-rod through the upper stinger, a wire of very high torsional stiffness. Each upper stinger provides two rotational degrees of freedom in bending. The opposite end of the push-rod is connected to the stator through the lower stinger, another short wire which allows three rotational degrees of freedom (two in bending, one in torsion) with respect to  $S$ .

Each stinger is modeled as a massless spring. The umbilicals are also considered to be massless; they are modeled together as a single, parallel spring-and-damper arrangement, attached at opposite ends to stator and flotor at effective umbilical attachment points  $S_u$  and  $F_u$ , respectively. This effective umbilical applies both a force and a moment to the flotor. The force is assumed to act at point  $F_u$ .

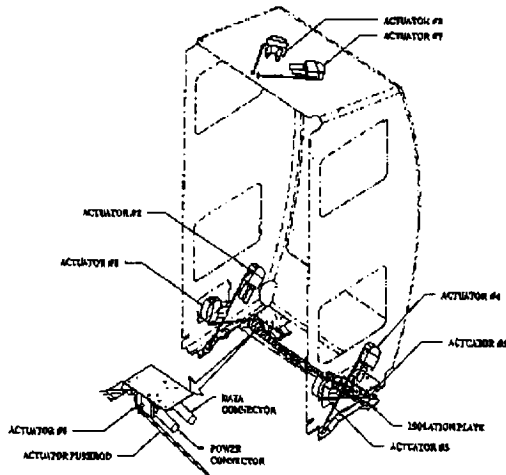


Figure 1. ARIS Control Assembly

The stator, the flotor, and each actuator arm and push-rod are considered to be rigid bodies, with mass centers at points  $S^*$ ,  $F^*$ ,  $A_i^*$ , and  $P_i^*$ , respectively. The superscript \* indicates the mass center of the indicated rigid body; the subscript  $i$  corresponds to the  $i^{th}$  actuator. ( $i = 1, \dots, 8$ ). All springs (cross-flexures and stingers) are assumed to be relaxed when the ISPR is centered in its rattlespace (the "home position").

## COORDINATE SYSTEMS

With the ISPR in the home position, fix eight right-handed, orthogonal coordinate systems in the flotor, one at each of the cross-flexure centers. Let the  $i^{th}$  coordinate system ( $i = 1, \dots, 8$ ) have origin  $F_i$  ( $i = 1, \dots, 8$ ) located at the center of the  $i^{th}$  cross-flexure, with axis directions determined by an orthonormal set of unit vectors  $\hat{f}_j^i$  ( $j = 1, 2, 3$ ). (The overhat indicates unit length, the index  $i$  corresponds to the  $i^{th}$  actuator assembly, and the index  $j$  distinguishes the three vectors.) Orient the unit vectors such that  $\hat{f}_2^i$  is along

the  $i^{th}$  arm, toward the  $i^{th}$  voice coil;  $\hat{f}_1^i$  is directed parallel to the other segment of the  $i^{th}$  arm and toward the upper stinger (which is located at  $A_2^i$ ); and  $\hat{f}_3^i$  is in the direction  $\hat{f}_1^i \times \hat{f}_2^i$  (along the intersection of the two cross-pieces of the  $i^{th}$  cross-flexure).

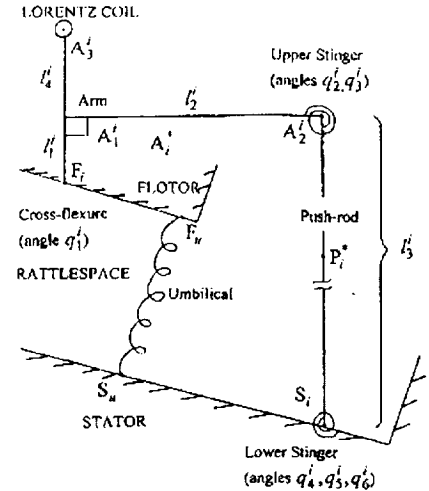


Figure 2. Kinematic Diagram, Including the  $i^{th}$  Actuator Assembly and the Umbilical

Fix a similar right-handed coordinate system  $\hat{a}_j^i$  ( $j = 1, 2, 3$ ) in the arm of each actuator. Locate each system  $\hat{a}_j^i$  such that it is coincident with the corresponding flotor-fixed coordinate system  $\hat{f}_j^i$  when the flotor is in the home position.

At the respective lower stingers (points  $S_i$ ), place eight push-rod-fixed coordinate systems  $\hat{p}_j^i$ , and eight stator-fixed coordinate

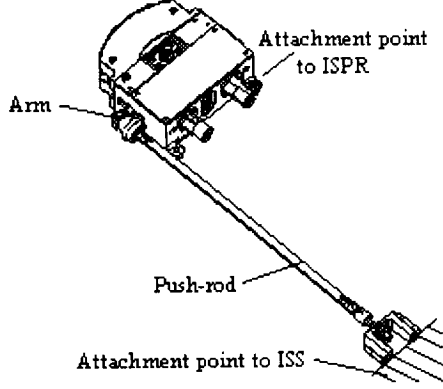


Figure 3. A Single ARIS Actuator

systems  $\underline{\hat{s}}_j^i$ . Orient these 24 coordinate systems such that when the stingers are relaxed (i.e., with the ISPR in the home position), the coordinate directions  $\underline{\hat{p}}_j^i$ , and  $\underline{\hat{s}}_j^i$  are co-aligned for the  $i^{\text{th}}$  actuator, with  $\underline{\hat{p}}_2^i$  (along with  $\underline{\hat{s}}_2^i$ , in the home position) directed from  $S_i$  toward  $A_2^i$ .

Define finally a primary, central, flotor-fixed, reference coordinate system with coordinate directions  $\underline{\hat{f}}_j$ . All other flotor-fixed coordinate systems  $\underline{\hat{f}}_j^i$  are assumed capable of being referenced (e.g., by known direction cosine angles) to this system. [See Equation (4).]

### ROTATION MATRICES

Let the  $\underline{\hat{a}}_j^i$  coordinate system rotate, relative to the  $\underline{\hat{f}}_j^i$  coordinate system, through positive angle  $q_1^i$  about the  $\underline{\hat{f}}_3^i$  axis. Similarly, let the orientation of the  $\underline{\hat{a}}_j^i$  coordinate system, relative to the  $\underline{\hat{p}}_j^i$  coordinate system, be described by consecutive positive rotations  $q_2^i$  (about the  $\underline{\hat{p}}_1^i$  axis) and  $q_3^i$  (about the moved 3-axis). And let the orientation of the  $\underline{\hat{p}}_j^i$  coordinate system, relative to the  $\underline{\hat{s}}_j^i$  coordinate system, be described by consecutive positive rotations  $q_4^i$  (about the  $\underline{\hat{s}}_3^i$  axis),  $q_5^i$  (about the moved 2-axis), and  $q_6^i$  (about the moved 1-axis).

Let  $c_j^i$  and  $s_j^i$  represent the cosines and sines of the respective angles  $q_j^i$ . Then the rotation matrices among the several coordinate systems for the  $i^{\text{th}}$  actuator assembly are as follows:

$$\begin{Bmatrix} \underline{\hat{p}}_1^i \\ \underline{\hat{p}}_2^i \\ \underline{\hat{p}}_3^i \end{Bmatrix} = \begin{bmatrix} c_4^i c_5^i & s_4^i c_5^i & -s_5^i \\ -s_4^i c_6^i + c_4^i s_5^i s_6^i & c_4^i c_6^i + s_4^i s_5^i s_6^i & c_5^i s_6^i \\ s_4^i s_6^i + c_4^i s_5^i c_6^i & -c_4^i s_6^i + s_4^i s_5^i c_6^i & c_5^i c_6^i \end{bmatrix} \begin{Bmatrix} \underline{\hat{s}}_1^i \\ \underline{\hat{s}}_2^i \\ \underline{\hat{s}}_3^i \end{Bmatrix}, \quad (1)$$

$$\begin{Bmatrix} \underline{\hat{a}}_1^i \\ \underline{\hat{a}}_2^i \\ \underline{\hat{a}}_3^i \end{Bmatrix} = \begin{bmatrix} c_3^i & c_2^i s_3^i & s_2^i s_3^i \\ -s_3^i & c_2^i c_3^i & s_2^i c_3^i \\ 0 & -s_2^i & c_2^i \end{bmatrix} \begin{Bmatrix} \underline{\hat{p}}_1^i \\ \underline{\hat{p}}_2^i \\ \underline{\hat{p}}_3^i \end{Bmatrix}, \quad (2)$$

and

$$\begin{Bmatrix} \underline{\hat{a}}_1^i \\ \underline{\hat{a}}_2^i \\ \underline{\hat{a}}_3^i \end{Bmatrix} = \begin{bmatrix} c_1^i & s_1^i & 0 \\ -s_1^i & c_1^i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underline{\hat{f}}_1^i \\ \underline{\hat{f}}_2^i \\ \underline{\hat{f}}_3^i \end{Bmatrix}. \quad (3)$$

Finally, define a rotation matrix between the eight flotor-fixed coordinate systems  $\underline{\hat{f}}_j^i$  and the single, flotor-fixed, reference

coordinate system  $\underline{\hat{f}}_j$ :

$$\begin{Bmatrix} \underline{\hat{f}}_1^i \\ \underline{\hat{f}}_2^i \\ \underline{\hat{f}}_3^i \end{Bmatrix} = \begin{bmatrix} f_{11}^i & f_{12}^i & f_{13}^i \\ f_{21}^i & f_{22}^i & f_{23}^i \\ f_{31}^i & f_{32}^i & f_{33}^i \end{bmatrix} \begin{Bmatrix} \underline{\hat{f}}_1 \\ \underline{\hat{f}}_2 \\ \underline{\hat{f}}_3 \end{Bmatrix}. \quad (4)$$

### GENERALIZED COORDINATES FOR $\tilde{S}$

The 48 angles  $q_j^i$  are the generalized coordinates of the system. For the  $i^{\text{th}}$  actuator the six associated generalized coordinates are as follows:  $q_1^i$  is the angle at the cross-flexure of the  $i^{\text{th}}$  actuator;  $q_2^i$  and  $q_3^i$  are the angles at the upper stinger; and  $q_4^i, q_5^i$  and  $q_6^i$  are the angles at the lower stinger.

### GENERALIZED SPEEDS FOR $\tilde{S}$

Define generalized speeds  $u_j^i$  for the system as the time rate of change of the generalized coordinates of  $\tilde{S}$  in the inertial reference frame:

$$u_j^i = \dot{q}_j^i \text{ (for } j=1, \dots, 6; i=1, \dots, 8). \quad (5)$$

### ANGULAR VELOCITIES OF REFERENCE FRAMES AND RIGID BODIES

Designate the reference frames corresponding to the stator, the  $i^{\text{th}}$  push rod, the  $i^{\text{th}}$  arm, and the flotor, by the symbols  $\hat{S}$ ,  $\hat{P}_i$ ,  $\hat{A}_i$ , and  $\hat{F}$ , respectively. Let  $\tilde{S}_i$  and  $\tilde{F}_i$  represent, respectively, the coordinate systems in  $\hat{S}$  and  $\hat{F}$  defined respectively by  $[\underline{\hat{s}}_1^i \ \underline{\hat{s}}_2^i \ \underline{\hat{s}}_3^i]^T$  and  $[\underline{\hat{f}}_1^i \ \underline{\hat{f}}_2^i \ \underline{\hat{f}}_3^i]^T$ . Two intermediate reference frames were introduced previously to permit describing the angular velocity of each push rod relative to the stator; designate those intermediate frames corresponding to the  $i^{\text{th}}$  actuator assembly by  $\tilde{R}_i$  and  $\tilde{Q}_i$ . Another intermediate reference frame was previously introduced between frames  $\hat{P}_i$  and  $\hat{A}_i$ ; designate this by  $\tilde{T}_i$ .

Let each intermediate reference frame have a frame-fixed, dextral set of unit vectors. Indicate the unit vectors for each of these frame-fixed coordinate systems by using the corresponding lower case letter ( $\underline{\hat{r}}_j^i$  corresponding to  $\tilde{R}_i$ , etc.). The following, then, give the

expressions for the angular velocities of the various reference frames and rigid bodies of  $\tilde{S}$  :

$${}^{F_i}\underline{\omega}^{A_i} = u_1^i \hat{f}_3^i, \quad {}^{P_i}\underline{\omega}^{T_i} = u_2^i \hat{p}_1^i, \quad {}^{T_i}\underline{\omega}^{A_i} = u_3^i \hat{t}_3^i \quad (6-8)$$

$${}^{S_i}\underline{\omega}^{R_i} = u_4^i \hat{s}_3^i, \quad {}^{R_i}\underline{\omega}^{Q_i} = u_5^i \hat{r}_2^i, \quad \text{and} \quad {}^{Q_i}\underline{\omega}^{P_i} = u_6^i \hat{q}_1^i. \quad (9-11)$$

Using the addition theorem for angular velocities, the angular velocities of the rigid bodies of  $\tilde{S}$  are

$${}^{S_i}\underline{\omega}^{A_i} = u_2^i \hat{p}_1^i + u_3^i \hat{t}_3^i + u_4^i \hat{s}_3^i + u_5^i \hat{r}_2^i + u_6^i \hat{q}_1^i, \quad (12)$$

$${}^{S_i}\underline{\omega}^{P_i} = u_4^i \hat{s}_3^i + u_5^i \hat{r}_2^i + u_6^i \hat{q}_1^i, \quad (13)$$

$$\text{and} \quad {}^{S_i}\underline{\omega}^{F_i} = u_2^i \hat{p}_1^i + u_3^i \hat{t}_3^i + u_4^i \hat{s}_3^i + u_5^i \hat{r}_2^i + u_6^i \hat{q}_1^i - u_1^i \hat{f}_3^i. \quad (14)$$

## BASIC ASSUMPTIONS

In the subsequent development of the ARIS equations of motion, it is assumed that ARIS works as intended; i.e., that the ARIS controller prevents the ISPR from exceeding its rattlespace constraints. It is also assumed that the small-angle approximations hold for angles  $q_j^i$ . Angular velocities and angular accelerations are assumed to be small as well. This means that the use of first-order linear perturbations will permit the full nonlinear equations of motion to be approximated accurately by a set of first-order linear differential equations. Finally, it is assumed that the angular velocity of the stator is negligible, and that the stator translational velocities and accelerations are small.

## LINEARIZED VELOCITIES OF THE CENTERS OF MASS FOR THE RIGID BODIES OF $\tilde{S}$

Represent by  $\underline{r}^{AB}$  the position vector from arbitrary point A to arbitrary point B. Define the following position vectors, using the indicated scalars:

$$\underline{r}^{F_i A_i} = l_1^i \hat{a}_2^i + l_2^i \hat{a}_1^i, \quad \underline{r}^{S_i A_i} = l_3^i \hat{p}_2^i, \quad \underline{r}^{S_i P_i} = p_2^i \hat{p}_2^i, \quad (15-17)$$

$$\underline{r}^{A_i A_i^*} = a_1^i \hat{a}_1^i + a_2^i \hat{a}_2^i, \quad \text{and} \quad \underline{r}^{F_i F_i^*} = f_1^i \hat{f}_1^i + f_2^i \hat{f}_2^i + f_3^i \hat{f}_3^i. \quad (18, 19)$$

First time derivatives of the appropriate position vectors, under the stated assumptions, yield expressions for the velocities of the centers of mass, for the seventeen rigid bodies. The following expressions are the linearized velocities for those centers of mass. (The pre-subscript indicates that the expressions are linearized; the pre-superscript indicates the reference frame assumed fixed for purposes of the differentiations.)

$${}^{S_i}\underline{v}^{P_i} = p_2^i (-u_4^i \hat{s}_1^i + u_6^i \hat{s}_3^i) \quad (i = 1, \dots, 8); \quad (20)$$

$${}^{S_i}\underline{v}^{A_i} = [-a_2^i u_3^i - (a_2^i + l_3^i) u_4^i] \hat{p}_1^i + (a_1^i u_3^i + a_1^i u_4^i) \hat{p}_2^i + [a_2^i u_2^i - a_1^i u_5^i + (a_2^i + l_3^i) u_6^i] \hat{p}_3^i, \quad (21)$$

$$\text{and} \quad {}^{S_i}\underline{v}^{F_i} = [f_2^i u_1^i - v_2^i u_3^i - (v_2^i + l_3^i) u_4^i + v_3^i u_5^i] \hat{p}_1^i + [-f_1^i u_1^i - v_3^i (u_2^i + u_6^i) + v_1^i (u_3^i + u_4^i)] \hat{p}_2^i + [v_2^i u_2^i - v_1^i u_5^i + (v_2^i + l_3^i) u_6^i] \hat{p}_3^i. \quad (22)$$

$$\text{where} \quad v_1^i = f_1^i - l_2^i, \quad v_2^i = f_2^i - l_1^i, \quad \text{and} \quad v_3^i = f_3^i. \quad (23-25)$$

## LINEARIZED ACCELERATIONS OF THE CENTERS OF MASS FOR THE RIGID BODIES OF $\tilde{S}$

Taking the time derivatives of the respective linearized velocity vectors yields expressions for the linearized accelerations of the centers of mass, for each rigid body. Note that the linearized velocity vectors may be used in this step—the full nonlinear accelerations need not be determined. This is a tremendous savings of effort, which would not be afforded if Newton's Second Law were applied directly, instead of Kane's approach.

$${}^{S_i}\underline{a}^{P_i} = p_2^i (-\dot{u}_4^i \hat{s}_1^i + \dot{u}_6^i \hat{s}_3^i), \quad (26)$$

$${}^{S_i}\underline{a}^{A_i} = [-a_2^i \dot{u}_3^i - (a_2^i + l_3^i) \dot{u}_4^i] \hat{p}_1^i + (a_1^i \dot{u}_3^i + a_1^i \dot{u}_4^i) \hat{p}_2^i + [a_2^i \dot{u}_2^i - a_1^i \dot{u}_5^i + (a_2^i + l_3^i) \dot{u}_6^i] \hat{p}_3^i, \quad (27)$$

$$\text{and} \quad {}^{S_i}\underline{a}^{F_i} = [f_2^i \dot{u}_1^i - v_2^i \dot{u}_3^i - (v_2^i + l_3^i) \dot{u}_4^i + v_3^i \dot{u}_5^i] \hat{p}_1^i + [-f_1^i \dot{u}_1^i - v_3^i (\dot{u}_2^i + \dot{u}_6^i) + v_1^i (\dot{u}_3^i + \dot{u}_4^i)] \hat{p}_2^i + [v_2^i \dot{u}_2^i - v_1^i \dot{u}_5^i + (v_2^i + l_3^i) \dot{u}_6^i] \hat{p}_3^i. \quad (28)$$

## LINEARIZED PARTIAL VELOCITY VECTORS FOR THE POINTS OF $\tilde{S}$ AT WHICH THE CONTACT/DISTANCE FORCES ARE ASSUMED TO ACT

The partial velocities and partial angular velocities are formed by inspection of the relevant velocity vectors. These partial velocities are then (and the order here is crucial) linearized by neglecting higher order terms.

### LINEARIZED PARTIAL VELOCITIES OF $P_i^*$

For the  $i^{\text{th}}$  push-rod the linearized partial velocities are

$${}^{S_i}\underline{v}_r^{P_i^*} = 0 \quad (\text{for } r=1,2,3), \quad {}^{S_i}\underline{v}_4^{P_i^*} = -p_2^i (\hat{s}_1^i + q_4^i \hat{s}_2^i), \quad (29, 30)$$

$${}^{S_i}\underline{v}_5^{P_i^*} = p_2^i q_6^i \hat{s}_1^i, \quad \text{and} \quad {}^{S_i}\underline{v}_6^{P_i^*} = p_2^i (q_5^i \hat{s}_1^i - q_6^i \hat{s}_2^i + \hat{s}_3^i). \quad (31, 32)$$

### LINEARIZED PARTIAL VELOCITIES OF $A_i^*$

For the  $i^{\text{th}}$  arm the linearized partial velocities are

$${}^{S_i}\underline{v}_1^{A_i^*} = 0, \quad {}^{S_i}\underline{v}_2^{A_i^*} = -a_2^i q_2^i \hat{p}_2^i + (a_2^i + a_1^i q_3^i) \hat{p}_3^i, \quad (33, 34)$$

$${}^{S_i}\underline{v}_3^{A_i^*} = -(a_2^i + a_1^i q_3^i) \hat{p}_1^i + (a_1^i - a_2^i q_3^i) \hat{p}_2^i + a_1^i q_2^i \hat{p}_3^i, \quad (35)$$

$${}^{S_i}\underline{v}_4^{A_i^*} = (a_2^i + a_1^i q_3^i) \hat{p}_1^i + (a_1^i - a_2^i q_3^i) \hat{p}_2^i - (a_2^i q_5^i + a_1^i q_6^i + l_3^i q_3^i) \hat{p}_3^i, \quad (36)$$

$${}^{S_i}\underline{v}_5^{A_i^*} = (a_2^i q_2^i + a_2^i q_6^i + l_3^i q_6^i) \hat{p}_1^i - a_1^i q_2^i \hat{p}_2^i + (a_2^i q_3^i - a_1^i) \hat{p}_3^i, \quad (37)$$

$$\text{and} \quad {}^{S_i}\underline{v}_6^{A_i^*} = -a_2^i q_2^i \hat{p}_2^i + (a_2^i + a_1^i q_2^i + l_3^i) \hat{p}_3^i. \quad (38)$$

### LINEARIZED PARTIAL VELOCITIES OF $F^*$

For the flotor, the linearized partial velocities are

$${}^{S_1}\underline{v}_1^{F^*} = f_2^1 \hat{f}_1^1 - f_1^1 \hat{f}_2^1, \quad (39)$$

$$S_{l2}^{F^*} = -(v_2^1 q_2^1 + v_3^1) \hat{p}_2^1 + (v_1^1 q_3^1 + v_2^1 - v_3^1 q_2^1 - f_1^1 q_1^1) \hat{p}_3^1, \quad (40)$$

$$S_{l3}^{F^*} = (-v_1^1 q_3^1 - v_2^1 + f_1^1 q_1^1) \hat{p}_1^1 + (v_1^1 - v_2^1 q_3^1 + f_2^1 q_1^1) \hat{p}_2^1 + v_1^1 q_2^1 \hat{p}_3^1, \quad (41)$$

$$S_{l4}^{F^*} = [-v_1^1 q_3^1 - v_2^1 + v_3^1 (q_2^1 + q_6^1) - f_1^1 q_1^1 - l_3^1] \hat{p}_1^1 + (v_1^1 - v_2^1 q_3^1 + v_3^1 q_5^1 + f_2^1 q_1^1) \hat{p}_2^1 - (v_1^1 q_6^1 + v_2^1 q_5^1 + l_3^1 q_5^1) \hat{p}_3^1, \quad (42)$$

$$S_{l5}^{F^*} = [v_2^1 (q_2^1 + q_6^1) + v_3^1 + l_3^1 q_6^1] \hat{p}_1^1 - v_1^1 q_6^1 \hat{p}_2^1 - (v_1^1 + v_2^1 q_3^1 + f_2^1 q_1^1) \hat{p}_3^1, \quad (43)$$

$$\text{and } S_{l6}^{F^*} = -(v_2^1 q_2^1 + v_3^1) \hat{p}_2^1 + (v_1^1 q_3^1 + v_2^1 - v_3^1 q_2^1 - f_1^1 q_1^1 + l_3^1) \hat{p}_3^1. \quad (44)$$

#### LINEARIZED PARTIAL VELOCITIES OF $F_u$

Define measure numbers for  $r^{F^* F_u}$  as follows:

$$r^{F^* F_u} = X_{F_u} \hat{f}_1^1 + Y_{F_u} \hat{f}_2^1 + Z_{F_u} \hat{f}_3^1. \quad (45)$$

$$\text{Since } S_{l r}^{F_u} = S_{l r}^{F^*} + \left[ \frac{\partial}{\partial u_r} \left( S_l \omega^F \times r^{F^* F_u} \right) \right], \quad (46)$$

the linearized partial velocities for the umbilical attachment point  $F_u$  can be expressed as follows:

$$S_{l1}^{F_u} = S_{l1}^{F^*} + Y_{F_u} \hat{f}_1^1 - X_{F_u} \hat{f}_2^1, \quad S_{l2}^{F_u} = S_{l2}^{F^*} - Z_{F_u} \hat{p}_2^1 + Y_{F_u} \hat{p}_3^1, \quad (47, 48)$$

$$S_{l3}^{F_u} = S_{l3}^{F^*} - Y_{F_u} \hat{p}_1^1 + X_{F_u} \hat{p}_2^1, \quad S_{l4}^{F_u} = S_{l4}^{F^*} - Y_{F_u} \hat{p}_1^1 + X_{F_u} \hat{p}_2^1, \quad (49, 50)$$

$$S_{l5}^{F_u} = S_{l5}^{F^*} + Z_{F_u} \hat{p}_1^1 - X_{F_u} \hat{p}_2^1, \quad \text{and } S_{l6}^{F_u} = S_{l6}^{F^*} - Z_{F_u} \hat{p}_2^1 + Y_{F_u} \hat{p}_3^1. \quad (51, 52)$$

#### LINEARIZED PARTIAL VELOCITIES OF $F_i$

The linearized partial velocities of  $F^i$  are

$$S_{l1}^{F_i} = 0, \quad S_{l2}^{F_i} = -(l_1^i + l_2^i q_3^i) \hat{a}_3^i, \quad S_{l3}^{F_i} = l_1^i \hat{a}_1^i - l_2^i \hat{a}_2^i, \quad (53-55)$$

$$S_{l4}^{F_i} = (l_1^i - l_3^i) \hat{a}_1^i - (l_3^i q_3^i + l_2^i) \hat{a}_2^i + [l_1^i q_5^i + l_2^i (q_2^i + q_6^i) + l_3^i q_5^i] \hat{a}_3^i, \quad (56)$$

$$S_{l5}^{F_i} = [-l_1^i (q_2^i + q_6^i) + l_3^i q_6^i] \hat{a}_1^i + l_2^i (q_2^i + q_6^i) \hat{a}_2^i + (l_2^i - l_1^i q_3^i) \hat{a}_3^i, \quad (57)$$

$$\text{and } S_{l6}^{F_i} = l_3^i q_2^i \hat{a}_2^i - (l_1^i + l_2^i q_3^i - l_3^i) \hat{a}_3^i. \quad (58)$$

#### LINEARIZED PARTIAL ANGULAR VELOCITIES FOR THE RIGID BODIES OF $\tilde{S}$

The following are the linearized partial angular velocities for the system.

$$\text{For the } i^{\text{th}} \text{ pushrod: } S_l \omega_r^P = 0 \quad (\text{for } r = 1, 2, 3), \quad (59)$$

$$S_l \omega_4^P = \hat{s}_3^i, \quad S_l \omega_5^P = \hat{r}_2^i, \quad \text{and } S_l \omega_6^P = \hat{q}_1^i. \quad (60-62)$$

$$\text{For the } i^{\text{th}} \text{ actuator arm: } S_l \omega_1^{A_i} = 0, \quad S_l \omega_2^{A_i} = \hat{p}_1^i, \quad (63, 64)$$

$$S_l \omega_3^{A_i} = \hat{r}_3^i, \quad S_l \omega_4^{A_i} = \hat{s}_3^i, \quad S_l \omega_5^{A_i} = \hat{r}_2^i, \quad \text{and } S_l \omega_6^{A_i} = \hat{q}_1^i \quad (65-68)$$

$$\text{For the flotor: } S_l \omega_1^F = -\hat{a}_1^i, \quad S_l \omega_2^F = \hat{p}_1^i, \quad S_l \omega_3^F = \hat{r}_3^i, \quad (69-71)$$

$$S_l \omega_4^F = \hat{s}_3^i, \quad S_l \omega_5^F = \hat{r}_2^i, \quad S_l \omega_6^F = \hat{q}_1^i \quad (72-74)$$

$$\text{and } S_l \omega_r^F = 0 \quad (r = 1, \dots, 6; i = 2, \dots, 8). \quad (75)$$

#### LINEARIZED ANGULAR ACCELERATIONS FOR THE RIGID BODIES OF $\tilde{S}$

##### LINEARIZED ANGULAR ACCELERATION OF ACTUATOR PUSH-ROD $\hat{P}_i$

$$S_l \alpha^{P_i} = \dot{u}_6^i \hat{p}_1^i + \dot{u}_5^i \hat{p}_2^i + \dot{u}_4^i \hat{p}_3^i. \quad (76)$$

##### LINEARIZED ANGULAR ACCELERATION OF ACTUATOR ARM $\hat{A}_i$

$$S_l \alpha^{A_i} = (\dot{u}_2^i + \dot{u}_6^i) \hat{a}_1^i + \dot{u}_5^i \hat{a}_2^i + (\dot{u}_3^i + \dot{u}_4^i) \hat{a}_3^i. \quad (77)$$

##### LINEARIZED ANGULAR ACCELERATION OF THE FLOTOR $\hat{F}$

$$S_l \alpha^F = (\dot{u}_2^i + \dot{u}_6^i) \hat{f}_1^i + \dot{u}_5^i \hat{f}_2^i + (\dot{u}_6^i - \dot{u}_1^i) \hat{f}_3^i. \quad (78)$$

#### CONTRIBUTIONS TO THE SET OF GENERALIZED ACTIVE FORCES DUE TO THE RIGID BODIES OF $\tilde{S}$

##### CONTRIBUTIONS DUE TO THE FLOTOR

On orbit (i.e., neglecting the effects of gravity), the flotor is acted upon by forces and moments due to the each Lorentz coil, actuator arm, and umbilical; and by direct disturbances.

Let  $-F^{C_i}$  and  $-M^{C_i}$  represent, respectively, the force and moment exerted by the  $i^{\text{th}}$  Lorentz coil (located at  $A_3^i$ ) on the flotor,

$$\text{where } F^{C_i} = F^{C_i} \hat{a}_1^i, \text{ assumed to act at point } F_i, \quad (79)$$

$$\text{and } M^{C_i} = r^{F_i A_3^i} \times F^{C_i} = -F^{C_i} (l_1^i + l_4^i) \hat{a}_3^i. \quad (80)$$

Let  $F^{F_i}$  and  $M^{F_i}$  represent, respectively, the force and moment exerted by the  $i^{\text{th}}$  actuator arm on the flotor, at the  $i^{\text{th}}$  cross-flexure. Since  $F^{F_i}$  is a noncontributing force, it can be ignored in the analysis.

The total moment  $M^F$  due to the eight cross-flexure springs has value

$$M^F = \sum_1^8 k_1^i q_1^i \hat{f}_3^i, \quad (81)$$

where  $k_1^i$  is the  $i^{\text{th}}$  cross-flexure spring stiffness.

Let  $F^U$  and  $M^U$  represent, respectively, the force and moment applied to the flotor by the umbilical; where the force is assumed to act at flotor-fixed point  $F_u$ . Umbilical force  $F^U$  is given by the equation

$$F^U = (-k_1 x_1 - c_1 \dot{x}_1) \hat{e}_1 + (-k_2 x_2 - c_2 \dot{x}_2) \hat{e}_2 + (-k_3 x_3 - c_3 \dot{x}_3) \hat{e}_3 + F_b; \quad (82)$$

where  $\hat{e}_i$  is some appropriate stator-fixed coordinate system;  $x_1, x_2$ , and  $x_3$  are the umbilical elongations in the respective  $\hat{e}_i$  ( $i = 1, 2, 3$ ) directions;  $F_b$  is the umbilical bias force in the home position;  $k_1$ ,

$k_2$ , and  $k_3$  are umbilical spring stiffnesses; and  $c_1$ ,  $c_2$ , and  $c_3$  are umbilical damping constants. Umbilical moment  $\underline{M}^U$  is given by  $\underline{M}^U = (-\kappa_1\phi_1 - \gamma_1\dot{\phi}_1)\hat{\underline{e}}_1 + (-\kappa_2\phi_2 - \gamma_2\dot{\phi}_2)\hat{\underline{e}}_2 + (-\kappa_3\phi_3 - \gamma_3\dot{\phi}_3)\hat{\underline{e}}_3 + \underline{M}_b$ , (83) where  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  are components of the umbilical angle of twist  $\phi$  in the respective  $\hat{\underline{e}}_i$  ( $i = 1, 2, 3$ ) directions;  $\underline{M}_b$  is the umbilical bias moment in the home position;  $\kappa_1$ ,  $\kappa_2$ , and  $\kappa_3$  are torsional umbilical spring stiffnesses; and  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_3$  are torsional umbilical damping constants.

Let  $\underline{F}^D$  and  $\underline{M}^D$  represent, respectively, the unknown disturbance force and moment acting on the flotor. Assume  $\underline{F}^D$  to act through the flotor mass center  $F^*$ . Define  $F_i^D$  and  $M_i^D$  to be the  $i^{\text{th}}$  components, respectively, of  $\underline{F}^D$  and  $\underline{M}^D$ , componentiated in  $\hat{\underline{F}}_1$ .

In terms of the above, the flotor's contribution to the set of generalized active forces, for the  $j^{\text{th}}$  generalized speed, is

$${}_i Q_r^{F_u} = {}_i \underline{v}_r^{F_u} \cdot \underline{F}^U + {}_i \underline{v}_r^{F^*} \cdot \underline{F}^D + {}_i \underline{v}_r^{F_i} \cdot (-\underline{F}^C) + {}_i \underline{\omega}_r^{F_i} \cdot (\underline{M}^U + \underline{M}^D - \underline{M}^C + \sum_1^8 k_1^i q_1^i \hat{f}_3^i). \quad (84)$$

The umbilical contributions to the  ${}_i Q_r^{F^*}$ 's, viz.,

${}_i \underline{v}_r^{F_u} \cdot \underline{F}^U + {}_i \underline{\omega}_r^{F^*} \cdot \underline{M}^U$ , are addressed in the following two sections.

The remaining terms of the  ${}_i Q_r^{F^*}$ 's are as follows.

$${}_i \underline{v}_1^{F^*} \cdot \underline{F}^D = f_2^1 F_1^D - f_1^1 F_2^D, \quad {}_i \underline{\omega}_1^{F^*} \cdot \underline{M}^D = -M_3^D, \quad (85, 86)$$

$${}_i \underline{v}_1^{F_i} \cdot (-\underline{F}^C) = 0, \quad {}_i \underline{\omega}_1^{F_i} \cdot (-\underline{M}^C) = -F^C (l_1^i + l_4^i), \quad (87, 88)$$

$${}_i \underline{\omega}_1^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right) = -k_1^i q_1^i - k_2^i q_2^i + k_3^i q_3^i + k_4^i q_4^i - k_5^i q_5^i, \quad (89)$$

$${}_i \underline{v}_2^{F^*} \cdot \underline{F}^D = v_3^1 (q_1^1 - q_3^1) F_1^D - v_3^1 F_2^D + F_3^D (v_1^1 q_3^1 + v_2^1 - f_1^1 q_1^1), \quad (90)$$

$${}_i \underline{v}_2^{F_i} \cdot \underline{M}^D = M_1^D + (q_1^1 - q_3^1) M_2^D, \quad {}_i \underline{v}_2^{F_i} \cdot (-\underline{F}^C) = 0, \quad (91, 92)$$

$${}_i \underline{\omega}_2^{F_i} \cdot (-\underline{M}^C) = 0, \quad {}_i \underline{\omega}_2^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right) = 0, \quad (93, 94)$$

$${}_i \underline{v}_3^{F^*} \cdot \underline{F}^D = (-v_1^1 q_1^1 - v_2^1 + f_1^1 q_1^1) F_1^D + (v_1^1 - v_2^1 q_1^1 + f_2^1 q_1^1) F_2^D, \quad (95)$$

$${}_i \underline{v}_3^{F_i} \cdot \underline{M}^D = M_3^D, \quad {}_i \underline{v}_3^{F_i} \cdot (-\underline{F}^C) = -F^C l_1^i, \quad (96, 97)$$

$${}_i \underline{\omega}_3^{F_i} \cdot (-\underline{M}^C) = F^C (l_1^i + l_4^i), \quad (98)$$

$${}_i \underline{\omega}_3^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right) = -k_1^i q_1^i - k_2^i q_2^i + k_3^i q_3^i + k_4^i q_4^i - k_5^i q_5^i, \quad (99)$$

$${}_i \underline{v}_4^{F^*} \cdot \underline{F}^D = [-l_3^1 - v_1^1 q_1^1 - v_2^1 + v_3^1 (q_2^1 + q_6^1) - f_1^1 q_1^1] F_1^D + [l_3^1 (q_3^1 - q_4^1 - q_1^1) + v_1^1 - v_2^1 q_1^1 + v_3^1 q_5^1 + f_2^1 q_1^1] F_2^D + [-v_1^1 (q_2^1 + q_6^1) - v_2^1 q_5^1] F_3^D, \quad (100)$$

$${}_i \underline{\omega}_4^{F^*} \cdot \underline{M}^D = (q_2^1 + q_6^1) M_2^D - q_5^1 M_1^D + M_3^D, \quad (101)$$

$${}_i \underline{v}_4^{F_i} \cdot (-\underline{F}^C) = -F^C (l_1^i - l_3^i), \quad {}_i \underline{\omega}_4^{F_i} \cdot (-\underline{M}^C) = {}_i \underline{\omega}_3^{F_i} \cdot (-\underline{M}^C), \quad (102, 103)$$

$${}_i \underline{\omega}_4^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right) = {}_i \underline{\omega}_3^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right), \quad (104)$$

$${}_i \underline{v}_5^{F^*} \cdot \underline{F}^D = [l_3^1 q_6^1 + v_2^1 (q_2^1 + q_6^1) + v_3^1] F_1^D + [-v_1^1 (q_2^1 + q_6^1) + v_3^1 (q_1^1 - q_3^1)] F_2^D + [-v_1^1 + v_2^1 q_3^1 - f_2^1 q_1^1] F_3^D, \quad (105)$$

$${}_i \underline{\omega}_5^{F^*} \cdot \underline{M}^D = (q_1^1 - q_3^1) M_1^D + M_2^D - (q_2^1 + q_6^1) M_3^D, \quad (106)$$

$${}_i \underline{v}_5^{F_i} \cdot (-\underline{F}^C) = F^C [l_1^i q_2^1 + (l_1^i - l_3^i) q_6^1], \quad (107)$$

$${}_i \underline{\omega}_5^{F_i} \cdot \underline{M}^C = F^C (l_1^i + l_4^i) (q_2^1 - q_6^1), \quad {}_i \underline{\omega}_5^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right) = 0, \quad (108, 109)$$

$${}_i \underline{v}_6^{F^*} \cdot \underline{F}^D = v_3^1 (q_1^1 - q_3^1) F_1^D + (l_3^1 q_2^1 - v_3^1) F_2^D + (l_3^1 + v_1^1 q_3^1 + v_2^1 - f_1^1 q_1^1) F_3^D, \quad (110)$$

$${}_i \underline{\omega}_6^{F^*} \cdot \underline{M}^D = {}_i \underline{\omega}_2^{F^*} \cdot \underline{M}^D, \quad {}_i \underline{v}_6^{F_i} \cdot (-\underline{F}^C) = -F^C l_3^i q_5^1, \quad (111, 112)$$

$${}_i \underline{\omega}_6^{F_i} \cdot (-\underline{M}^C) = 0, \quad \text{and } {}_i \underline{\omega}_6^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right) = {}_i \underline{\omega}_5^{F_i} \cdot \left( \sum_1^8 k_1^i q_1^i \hat{f}_3^i \right). \quad (113, 114)$$

Notice the coupling between the unknown-disturbance measure numbers and the generalized coordinates. This coupling will make the disturbance input matrix  $E$  [in Eq. (203)] time-varying.

### UMBILICAL FORCE $\underline{F}^U$

Equation (82) expresses umbilical force  $\underline{F}^U$  in terms of umbilical-elongation components  $x_1$ ,  $x_2$ , and  $x_3$ , and their time derivatives. These items must be re-expressed in terms of the generalized coordinates and generalized speeds.

If the umbilical attachment point  $F_u$  is at  $F_{uh}$  in the home position, then

$$x_i = (\underline{r}^{S_u F_u} - \underline{r}^{S_u F_{uh}}) \cdot \hat{\underline{e}}_i, \quad \text{for } i = 1, 2, 3. \quad (115)$$

$$\text{But } \underline{r}^{S_u F_u} - \underline{r}^{S_u F_{uh}} = \underline{r}^{F_{uh} F_u} = \underline{r}^{S_i F_i} + \underline{r}^{F_i F_u} - \underline{r}^{S_i S_u} - \underline{r}^{S_u F_{uh}}, \quad (116a, b)$$

where the right-hand-side terms can be expressed by

$$\underline{r}^{S_i F_i} = l_3^i \hat{p}_2^1 - l_2^i \hat{a}_1^1 - l_1^i \hat{a}_2^1, \quad \underline{r}^{F_i F_u} = x_{F_u} \hat{f}_1^1 + y_{F_u} \hat{f}_2^1 + z_{F_u} \hat{f}_3^1 \quad (117, 118)$$

$$\underline{r}^{S_i S_u} = x_{S_u} \hat{s}_1^1 + y_{S_u} \hat{s}_2^1 + z_{S_u} \hat{s}_3^1, \quad \text{and } \underline{r}^{S_u F_{uh}} = x_0 \hat{s}_1^1 + y_0 \hat{s}_2^1 + z_0 \hat{s}_3^1, \quad (119, 120)$$

for appropriately defined coefficients.

Define now the following rotation matrix:

$$\begin{Bmatrix} \hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} \hat{\underline{s}}_1^1 \\ \hat{\underline{s}}_2^1 \\ \hat{\underline{s}}_3^1 \end{Bmatrix}. \quad (121)$$

In terms of the  $\hat{\underline{e}}_i$  coordinate system, Eq. (116b) can now be written as

$$\underline{r}^{F_{uh} F_u} = x_1 \hat{\underline{e}}_1 + x_2 \hat{\underline{e}}_2 + x_3 \hat{\underline{e}}_3, \quad (122)$$

$$\text{where } \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} C_1 + y_{F_u} (q_1^1 - q_3^1 - q_4^1) + z_{F_u} q_5^1 - l_3^1 q_4^1 + l_1^1 (q_3^1 + q_4^1) \\ C_2 - z_{F_u} (q_2^1 + q_6^1) - x_{F_u} (q_1^1 - q_3^1 - q_4^1) - l_2^1 (q_3^1 + q_4^1) \\ C_3 + l_3^1 - x_{F_u} q_5^1 + y_{F_u} - l_1^1 (q_2^1 + q_6^1) + l_3^1 q_6^1 \end{Bmatrix} \quad (123)$$

for

$$C_1 = x_{F_u} - x_{S_u} - l_2^1 - x_0, \quad (124)$$

$$C_2 = y_{F_u} - y_{S_u} + l_3^1 - l_1^1 - y_0, \quad \text{and } C_3 = z_{F_u} - z_{S_u} - z_0. \quad (125, 126)$$

Differentiating,

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \cdot \begin{Bmatrix} y_{F_u}(u_1^1 - u_3^1 - u_4^1) + z_{F_u} u_5^1 - l_3^1 u_4^1 + l_1^1(u_3^1 + u_4^1) \\ -z_{F_u}(u_2^1 + u_6^1) - x_{F_u}(u_1^1 - u_3^1 - u_4^1) - l_2^1(u_3^1 + u_4^1) \\ (l_2^1 - x_{F_u})u_5^1 + (y_{F_u} - l_1^1)(u_2^1 + u_6^1) + l_3^1 u_6^1 \end{Bmatrix} \quad (127)$$

Using Eqs. (82) and (123) through (127), a linearized expression could now be written straightforwardly for the umbilical force  $\underline{F}^U$ .

### UMBILICAL MOMENT $\underline{M}^U$

Equation (83) expresses umbilical moment  $\underline{M}^U$  in terms of angle-of-twist components  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$ , and their time derivatives. These items must be re-expressed in terms of the generalized coordinates and generalized speeds.

Let  $\phi \hat{n}_\phi$  represent the rotation of the flotor, relative to the stator, from the home position.  $\hat{n}_\phi$  is the rotation axis, and  $\phi$  is the angle of twist about that axis. Note that  $\phi_i = \phi \hat{n}_\phi \cdot \hat{e}_i$ , for  $i=1,2,3$ . (128)

$$\text{Express } \hat{n}_\phi \text{ as } \hat{n}_\phi = g_1 \hat{f}_1^1 + g_2 \hat{f}_2^1 + g_3 \hat{f}_3^1. \quad (129)$$

$$\text{Define rotation matrix } Q \text{ by } \begin{Bmatrix} \hat{f}_1^1 \\ \hat{f}_2^1 \\ \hat{f}_3^1 \end{Bmatrix} = [Q] \begin{Bmatrix} \hat{s}_1^1 \\ \hat{s}_2^1 \\ \hat{s}_3^1 \end{Bmatrix}. \quad (130)$$

The linearized 3x3 rotation matrix  ${}_I Q$  has elements  ${}_I Q_{ij}$  defined as follows:

$$\begin{Bmatrix} \hat{f}_1^1 \\ \hat{f}_2^1 \\ \hat{f}_3^1 \end{Bmatrix} = \begin{bmatrix} 1 & -q_1^1 + q_3^1 + q_4^1 & -q_5^1 \\ q_1^1 - q_3^1 - q_4^1 & 1 & q_2^1 + q_6^1 \\ q_5^1 & -q_2^1 - q_6^1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{s}_1^1 \\ \hat{s}_2^1 \\ \hat{s}_3^1 \end{Bmatrix} \quad (131)$$

For small  $\phi$  it can be shown (Salcudean, 1991) that

$$\phi \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix} = \frac{{}_I Q - {}_I Q^T}{(1 + \text{tr } {}_I Q)^{1/2}}, \quad (132)$$

where the post-superscript  $T$  indicates matrix transposition and  $\text{tr } {}_I Q$  represents the trace of  ${}_I Q$ . Substitution from Eq. (131) into Eq.

$$(129), \text{ and simplification, yields } g_1 = -\frac{1}{\phi} \cdot (q_2^1 + q_6^1), \quad (133)$$

$$g_2 = -\frac{1}{\phi} \cdot q_5^1, \quad \text{and} \quad g_3 = \frac{1}{\phi} \cdot (q_1^1 - q_3^1 - q_4^1). \quad (134, 135)$$

Substituting from Eqs. (133)-(135) into Eq. (129), and transforming into the  $\hat{s}_i^1$  coordinate system by use of  ${}_I Q$ , one obtains the following expression for the spin axis:

$${}_I \hat{n}_\phi = \frac{1}{\phi} \left[ -(q_2^1 + q_6^1) \hat{s}_1^1 - q_5^1 \hat{s}_2^1 + (q_1^1 - q_3^1 - q_4^1) \hat{s}_3^1 \right]. \quad (136)$$

$$\text{Since } {}_I \hat{n}_\phi \text{ has unit length, } {}_I \phi = \left[ (q_2^1 + q_6^1)^2 + (q_5^1)^2 + (q_1^1 - q_3^1 - q_4^1)^2 \right]^{1/2}. \quad (137)$$

Use of Eqs. (128), and (136) leads to the following linearized forms for angular position and rotation rate:

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} -(q_2^1 + q_6^1) \\ q_5^1 \\ q_1^1 - q_3^1 - q_4^1 \end{Bmatrix}, \quad (138)$$

$$\text{and} \quad \begin{Bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{Bmatrix} -(u_2^1 + u_6^1) \\ u_5^1 \\ u_1^1 - u_3^1 - u_4^1 \end{Bmatrix}. \quad (139)$$

From Eqs. (84), (138), and (139), a linearized expression could now be written straightforwardly for the umbilical moment  $\underline{M}^U$ . The flotor's contribution to the set of generalized active forces, for the  $r^{\text{th}}$  generalized speed, could then be found by substituting the expressions for  $\underline{F}^U$  (previous section) and  $\underline{M}^U$ , into Eq. (84).

### CONTRIBUTIONS DUE TO THE ACTUATOR ARMS

The forces and moments acting on the  $i^{\text{th}}$  actuator arm are due to the respective Lorentz coil (located at  $A_3^i$ ), the flotor (through the  $i^{\text{th}}$  cross-flexure), and the respective push-rod (through the upper stinger). The coil force  $\underline{F}^{C_i}$  is the only contributing force. The contributing loads, in the above indicated order, are as follows:

$$\underline{F}^{C_i} = F^{C_i} \hat{a}_1^i, \text{ assumed to act at point } F_i, \quad (140)$$

$$\underline{M}^{C_i} = \underline{r}^{F_i A_3^i} \times \underline{F}^{C_i} = (l_1^i + l_4^i) \hat{a}_2^i \times F^{C_i} \hat{a}_1^i = -F^{C_i} (l_1^i + l_4^i) \hat{a}_3^i, \quad (141)$$

$$-\underline{M}^{F_i} = -k_1^i q_1^i \hat{f}_3^i, \text{ and } \underline{M}^{A_i} = -k_2^i q_2^i \hat{p}_1^i - k_3^i q_3^i \hat{f}_3^i, \quad (142, 143)$$

where  $l_2^i$  and  $l_4^i$  are pertinent geometric lengths, and  $k_2^i$  and  $k_3^i$  are pertinent upper-stinger spring stiffnesses.

In terms of the above, the contribution for the  $i^{\text{th}}$  actuator arm to the set of generalized active forces, for the  $r^{\text{th}}$  generalized speed, is

$${}_I Q_r^{A_i} := {}_I \underline{v}_r^{A_i} \cdot \underline{F}^{C_i} + {}_I \underline{\omega}_r^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}). \quad (144)$$

The individual terms of the  ${}_I \underline{\omega}_r^{A_i}$ 's are as follows:

$${}_I \underline{v}_1^{F_i} \cdot \underline{F}^{C_i} = 0, \quad {}_I \underline{\omega}_1^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = 0, \quad (145, 146)$$

$${}_I \underline{v}_2^{F_i} \cdot \underline{F}^{C_i} = 0, \quad {}_I \underline{\omega}_2^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -k_2^i q_2^i, \quad (147, 148)$$

$${}_I \underline{v}_3^{F_i} \cdot \underline{F}^{C_i} = F^{C_i} l_1^i, \quad (149)$$

$${}_I \underline{\omega}_3^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = k_1^i q_1^i - k_3^i q_3^i - F^{C_i} (l_1^i + l_4^i), \quad (150)$$

$${}_I \underline{v}_4^{F_i} \cdot \underline{F}^{C_i} = F^{C_i} (l_1^i - l_3^i), \quad (151)$$

$${}_I \underline{\omega}_4^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -k_3^i q_3^i - F^{C_i} (l_1^i + l_4^i), \quad (152)$$

$${}_I \underline{v}_5^{F_i} \cdot \underline{F}^{C_i} = -F^{C_i} [l_1^i (q_2^i + q_6^i) - l_3^i q_6^i], \quad (153)$$

$${}_I \underline{\omega}_5^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -F^{C_i} (l_1^i + l_4^i) (q_2^i + q_6^i), \quad (154)$$

$${}_I \underline{v}_6^{A_i} \cdot \underline{F}^{C_i} = 0, \quad \text{and} \quad {}_I \underline{\omega}_6^{A_i} \cdot (\underline{M}^{A_i} + \underline{M}^{C_i} - \underline{M}^{F_i}) = -k_2^i q_2^i. \quad (155, 156)$$



Notice the coupling between the control inputs and the generalized coordinates. This coupling will make the disturbance input matrix  $B$  [in Eq. (204)] time-varying.

#### CONTRIBUTIONS DUE TO THE PUSH-RODS

The contributing loads on each push-rod are moments  $\underline{M}^{P_i}$  and  $-\underline{M}^{A_i}$ , where (using pertinent lower-stinger stiffnesses)

$$\underline{M}^{P_i} = -k_4^i q_4^i \hat{s}_3^i - k_5^i q_5^i \hat{s}_2^i - k_6^i q_6^i \hat{q}_1^i. \quad (157)$$

The contribution for the  $i^{\text{th}}$  push-rod to the set of generalized active forces, for the  $r^{\text{th}}$  generalized speed, is

$${}_i Q_r^{P_i} = {}_i \underline{\omega}_r^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}). \quad (158)$$

The individual terms of the  ${}_i Q_r^{P_i}$ 's are as follows:

$${}_i \underline{\omega}_r^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = 0 \quad (r = 1, 2, 3), \quad (159)$$

$${}_i \underline{\omega}_4^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = k_3^i q_3^i - k_4^i q_4^i, \quad (160)$$

$${}_i \underline{\omega}_5^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = -k_5^i q_5^i, \quad (161)$$

$$\text{and } {}_i \underline{\omega}_6^{P_i} \cdot (\underline{M}^{P_i} - \underline{M}^{A_i}) = k_2^i q_2^i - k_6^i q_6^i. \quad (162)$$

#### CONTRIBUTIONS TO THE SET OF GENERALIZED INERTIA FORCES DUE TO THE RIGID BODIES OF $\tilde{S}$

Represent by  $I_{jk}^{A_i/A_i^*}$  the central moment/product of inertia of the  $i^{\text{th}}$  actuator arm for the  $j$  and  $k$  body-fixed coordinate directions  $\hat{a}_j^i$  and  $\hat{a}_k^i$ . Define push-rod inertias  $I_j^{P_i/P_i^*}$  analogously, where the single subscript indicates that the axes are assumed to be principal axes. Let  $I_{jk}^{F/F^*}$  represent the central inertia scalar of the flotor for the flotor-fixed coordinate directions  $\hat{f}_j^1$  and  $\hat{f}_k^1$ . Use the symbol  $\underline{H}$  to represent an angular momentum vector. Associated post-superscripts on  $\underline{H}$  have the same meanings as for the inertias. The contributions to the generalized inertia forces for  $\tilde{S}$  can now be expressed.

#### CONTRIBUTIONS DUE TO THE PUSH-RODS

The contributions  $({}_i Q_r^*)^{P_i}$  to the generalized inertia forces due to the  $i^{\text{th}}$  push-rod are as follows:

$$({}_i Q_r^*)^{P_i} = {}_i \underline{\omega}_r^{P_i^*} \cdot \left( -m_{P_i} {}_i \underline{a}^{P_i^*} \right) + {}_i \underline{\omega}_r^{P_i} \cdot \left( -{}_i \underline{H}^{P_i/P_i^*} \right) = 0 \quad (163)$$

(for  $r = 1, 2, 3; i = 1, \dots, 8$ ),

$$({}_i Q_4^*)^{P_i} = {}_i \underline{\omega}_4^{P_i^*} \cdot \left( -m_{P_i} {}_i \underline{a}^{P_i^*} \right) + {}_i \underline{\omega}_4^{P_i} \cdot \left( -{}_i \underline{H}^{P_i/P_i^*} \right) = -\left[ m_{P_i} (p_2^i)^2 + I_3^{P_i/P_i^*} \right] \dot{u}_4^i, \quad (164)$$

$$({}_i Q_5^*)^{P_i} = {}_i \underline{\omega}_5^{P_i^*} \cdot \left( -m_{P_i} {}_i \underline{a}^{P_i^*} \right) + {}_i \underline{\omega}_5^{P_i} \cdot \left( -{}_i \underline{H}^{P_i/P_i^*} \right) = -I_2^{P_i/P_i^*} \dot{u}_5^i, \quad (165)$$

$$\begin{aligned} ({}_i Q_6^*)^{P_i} &= {}_i \underline{\omega}_6^{P_i^*} \cdot \left( -m_{P_i} {}_i \underline{a}^{P_i^*} \right) + {}_i \underline{\omega}_6^{P_i} \cdot \left( -{}_i \underline{H}^{P_i/P_i^*} \right) \\ &= -\left[ m_{P_i} (p_2^i)^2 + I_1^{P_i/P_i^*} \right] \dot{u}_6^i. \end{aligned} \quad (166)$$

#### CONTRIBUTIONS DUE TO THE ACTUATOR ARMS

The contribution  $({}_i Q_r^*)^{A_i}$  to the generalized inertia forces due to the  $i^{\text{th}}$  actuator arm, for the  $r^{\text{th}}$  generalized speed, is

$$({}_i Q_r^*)^{A_i} = {}_i \underline{\omega}_r^{A_i^*} \cdot \left( -m_{A_i} {}_i \underline{a}^{A_i^*} \right) + {}_i \underline{\omega}_r^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right). \quad (167)$$

Then the individual, nonzero terms of the  $({}_i Q_r^*)^{A_i}$ 's are as follows:

$${}_i \underline{\omega}_2^{A_i^*} \cdot \left( -m_{A_i} {}_i \underline{a}^{A_i^*} \right) = -m_{A_i} \left\{ (a_2^i)^2 \dot{u}_2^i + [a_2^i (a_2^i + l_3^i)] \dot{u}_6^i - a_1^i a_2^i \dot{u}_5^i \right\}, \quad (168)$$

$${}_i \underline{\omega}_2^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right) = -I_{11}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{12}^{A_i/A_i^*} \dot{u}_5^i - I_{13}^{A_i/A_i^*} (\dot{u}_3^i + \dot{u}_4^i), \quad (169)$$

$$\begin{aligned} {}_i \underline{\omega}_3^{A_i^*} \cdot \left( -m_{A_i} {}_i \underline{a}^{A_i^*} \right) &= -m_{A_i} \left\{ [(a_1^i)^2 + (a_2^i)^2] \dot{u}_3^i \right. \\ &\quad \left. + [(a_1^i)^2 + a_2^i (a_2^i + l_3^i)] \dot{u}_4^i \right\} \end{aligned} \quad (170)$$

$${}_i \underline{\omega}_3^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right) = -I_{31}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{32}^{A_i/A_i^*} \dot{u}_5^i - I_{33}^{A_i/A_i^*} (\dot{u}_3^i + \dot{u}_4^i), \quad (171)$$

$$\begin{aligned} {}_i \underline{\omega}_4^{A_i^*} \cdot \left( -m_{A_i} {}_i \underline{a}^{A_i^*} \right) &= -m_{A_i} \left\{ [(a_1^i)^2 + a_2^i (a_2^i + l_3^i)] \dot{u}_3^i \right. \\ &\quad \left. + [(a_1^i)^2 + (a_2^i + l_3^i)^2] \dot{u}_4^i \right\}, \end{aligned} \quad (172)$$

$${}_i \underline{\omega}_4^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right) = {}_i \underline{\omega}_3^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right), \quad (173)$$

$$\begin{aligned} {}_i \underline{\omega}_5^{A_i^*} \cdot \left( -m_{A_i} {}_i \underline{a}^{A_i^*} \right) &= -m_{A_i} \left\{ [(a_1^i)^2] \dot{u}_5^i \right. \\ &\quad \left. - a_1^i a_2^i \dot{u}_2^i - a_1^i (a_2^i + l_3^i) \dot{u}_6^i \right\}, \end{aligned} \quad (174)$$

$${}_i \underline{\omega}_5^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right) = -I_{21}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i) - I_{22}^{A_i/A_i^*} \dot{u}_5^i - I_{23}^{A_i/A_i^*} (\dot{u}_2^i + \dot{u}_6^i), \quad (175)$$

$$\begin{aligned} {}_i \underline{\omega}_6^{A_i^*} \cdot \left( -m_{A_i} {}_i \underline{a}^{A_i^*} \right) &= -m_{A_i} \left\{ a_2^i (a_2^i + l_3^i) \dot{u}_2^i \right. \\ &\quad \left. + (a_2^i + l_3^i)^2 \dot{u}_6^i - a_1^i (a_2^i + l_3^i) \dot{u}_5^i \right\}, \end{aligned} \quad (176)$$

$$\text{and } {}_i \underline{\omega}_6^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right) = {}_i \underline{\omega}_2^{A_i} \cdot \left( -{}_i \underline{H}^{A_i/A_i^*} \right). \quad (177)$$

#### CONTRIBUTIONS DUE TO THE FLOTOR

The contribution  $({}_i Q_r^*)^F$  to the generalized inertia forces, for the  $r^{\text{th}}$  generalized speed, is

$$({}_i Q_r^*)^F = {}_i \underline{\omega}_r^{F^*} \cdot \left( -m_F {}_i \underline{a}^{F^*} \right) + {}_i \underline{\omega}_r^F \cdot \left( -{}_i \underline{H}^{F/F^*} \right). \quad (178)$$

Then the individual terms of the  $({}_l Q_r^*)^F$ 's are

$$S_{l1}^{F^*} \cdot \left( -m_F S_{l1} \underline{a}^{F^*} \right) = -m_F \left\{ \left[ (f_2^1)^2 + (f_1^1)^2 \right] \dot{u}_1^1 + f_1^1 v_3^1 \dot{u}_2^1 \right. \\ \left. + \left[ -f_1^1 v_1^1 - f_2^1 v_2^1 \right] \dot{u}_3^1 - \left[ f_2^1 (v_2^1 + l_3^1) + f_1^1 v_1^1 \right] \dot{u}_4^1 + f_2^1 v_3^1 \dot{u}_5^1 + f_1^1 v_3^1 \dot{u}_6^1 \right\}, \quad (179)$$

$$S_{l2}^{F^*} \cdot \left( -I_{31}^{F/F^*} \dot{u}_2^1 + I_{32}^{F/F^*} \dot{u}_5^1 + I_{33}^{F/F^*} (\dot{u}_3^1 + \dot{u}_4^1 - \dot{u}_6^1) \right), \quad (180)$$

$$S_{l2}^{F^*} \cdot \left( -m_F S_{l2} \underline{a}^{F^*} \right) = -m_F \left\{ v_1^1 f_1^1 \dot{u}_1^1 + \left[ (v_1^1)^2 + (v_3^1)^2 \right] \dot{u}_2^1 - v_1^1 v_3^1 \dot{u}_3^1 - v_1^1 v_3^1 \dot{u}_4^1 \right. \\ \left. - v_1^1 v_2^1 \dot{u}_5^1 + \left[ (v_3^1)^2 + v_2^1 (v_2^1 + l_3^1) \right] \dot{u}_6^1 \right\}, \quad (181)$$

$$S_{l2}^{F^*} \cdot \left( -I_{11}^{F/F^*} (\dot{u}_2^1 + \dot{u}_6^1) - I_{12}^{F/F^*} \dot{u}_5^1 - I_{13}^{F/F^*} (\dot{u}_3^1 + \dot{u}_4^1 - \dot{u}_6^1) \right), \quad (182)$$

$$S_{l2}^{F^*} \cdot \left( -m_F S_{l2} \underline{a}^{F^*} \right) = -m_F \left\{ \left[ -f_2^1 v_1^1 - f_1^1 v_1^1 \right] \dot{u}_1^1 - v_1^1 v_3^1 \dot{u}_2^1 + \left[ (v_1^1)^2 + (v_3^1)^2 \right] \dot{u}_3^1 + \left[ (v_1^1)^2 + v_2^1 (v_2^1 + l_3^1) \right] \dot{u}_4^1 - v_2^1 v_3^1 \dot{u}_5^1 - v_1^1 v_3^1 \dot{u}_6^1 \right\}, \quad (183)$$

$$S_{l3}^{F^*} \cdot \left( -I_{31}^{F/F^*} (\dot{u}_2^1 + \dot{u}_6^1) - I_{32}^{F/F^*} \dot{u}_5^1 - I_{33}^{F/F^*} (\dot{u}_3^1 + \dot{u}_4^1 - \dot{u}_6^1) \right), \quad (184)$$

$$S_{l3}^{F^*} \cdot \left( -m_F S_{l3} \underline{a}^{F^*} \right) = -m_F \left\{ -f_2^1 (l_3^1 + v_2^1) \dot{u}_1^1 - f_1^1 v_1^1 \dot{u}_1^1 - v_1^1 v_3^1 \dot{u}_2^1 + \left[ (v_1^1)^2 + v_2^1 (v_2^1 + l_3^1) \right] \dot{u}_3^1 \right. \\ \left. + \left[ (v_1^1)^2 + (v_2^1 + l_3^1)^2 \right] \dot{u}_4^1 - v_3^1 (v_2^1 + l_3^1) \dot{u}_5^1 - v_1^1 v_3^1 \dot{u}_6^1 \right\}, \quad (185)$$

$$S_{l4}^{F^*} \cdot \left( -I_{11}^{F/F^*} \dot{u}_2^1 + I_{12}^{F/F^*} \dot{u}_5^1 + I_{13}^{F/F^*} (\dot{u}_3^1 + \dot{u}_4^1 - \dot{u}_6^1) \right), \quad (186)$$

$$S_{l4}^{F^*} \cdot \left( -m_F S_{l4} \underline{a}^{F^*} \right) = -m_F \left\{ f_2^1 v_3^1 \dot{u}_1^1 - v_1^1 v_2^1 \dot{u}_2^1 - v_2^1 v_3^1 \dot{u}_3^1 - v_3^1 (v_2^1 + l_3^1) \dot{u}_4^1 + \left[ (v_1^1)^2 + (v_3^1)^2 \right] \dot{u}_5^1 - v_1^1 (v_2^1 + l_3^1) \dot{u}_6^1 \right\}, \quad (187)$$

$$S_{l5}^{F^*} \cdot \left( -I_{21}^{F/F^*} (\dot{u}_2^1 + \dot{u}_6^1) - I_{22}^{F/F^*} \dot{u}_5^1 - I_{23}^{F/F^*} (\dot{u}_3^1 + \dot{u}_4^1 - \dot{u}_6^1) \right), \quad (188)$$

$$S_{l5}^{F^*} \cdot \left( -m_F S_{l5} \underline{a}^{F^*} \right) = -m_F \left\{ f_1^1 v_3^1 \dot{u}_1^1 + (v_3^1)^2 \dot{u}_2^1 + v_2^1 (v_2^1 + l_3^1) \dot{u}_3^1 - v_1^1 v_3^1 \dot{u}_4^1 - v_1^1 v_3^1 \dot{u}_5^1 \right. \\ \left. - v_1^1 (v_2^1 + l_3^1) \dot{u}_6^1 + \left[ (v_2^1 + l_3^1)^2 + (v_3^1)^2 \right] \dot{u}_6^1 \right\}, \quad (189)$$

$$\text{and } S_{l6}^{F^*} \cdot \left( -I_{11}^{F/F^*} \dot{u}_2^1 + I_{12}^{F/F^*} \dot{u}_5^1 + I_{13}^{F/F^*} (\dot{u}_3^1 + \dot{u}_4^1 - \dot{u}_6^1) \right), \quad (190)$$

## EQUATIONS OF MOTION FOR THE SYSTEM

### KINEMATICAL EQUATIONS

There are 48 kinematical equations for the system, one for each generalized speed:  $u_j^i = \dot{q}_j^i$  (for  $j = 1, \dots, 6$ ;  $i = 1, \dots, 8$ ). (191)

### DYNAMICAL EQUATIONS

Six dynamical equations are obtained using the following process. First, add the respective contributions of the 17 rigid bodies to the set of holonomic generalized active and holonomic generalized inertia forces, for each generalized speed (i.e.,  $r = 1, \dots, 48$ ). The holonomic generalized active force for the  $r^{\text{th}}$  generalized speed is

$$F_r = \sum_{j=1}^{17} (F_r)^j, \quad (192)$$

where  $(F_r)^j$  is the contribution to the set of holonomic generalized active forces due to the  $j^{\text{th}}$  rigid body. That is,

$$F_r = Q_r^F + \sum_{i=1}^8 Q_r^{S_i} + \sum_{i=1}^8 Q_r^{P_i}. \quad (193)$$

Likewise the contribution to the set of holonomic generalized inertia forces is

$$F_r^* = \sum_{j=1}^{17} (F_r^*)^j = ({}_l Q_r^*)^F + \sum_{i=1}^8 ({}_l Q_r^*)^{S_i} + \sum_{i=1}^8 ({}_l Q_r^*)^{P_i}. \quad (194)$$

Second, develop the relationship between the dependent and the independent generalized speeds in the form:

$$u_j^i = \sum_{s=1}^6 A_{rs} u_s^1 + B_r \quad (i = 2, \dots, 8; j = 1, \dots, 6; r = 7, \dots, 48). \quad (195)$$

Where in the above equation  $u_s^1$  are the six independent generalized speeds, and  $u_j^i$  are the 42 dependent generalized speeds.  $A_{rs}$  is a  $42 \times 6$  matrix, derived from the nonholonomic constraint equations (see next section). The nonholonomic and holonomic generalized active forces are related to each other as follows:

$$\tilde{F}_r = F_r + \sum_{s=7}^{48} F_s A_{sr} \quad (r = 1, \dots, 6). \quad (196)$$

Similarly, the nonholonomic and holonomic generalized inertial forces are related to each other as follows:

$$\tilde{F}_r^* = F_r^* + \sum_{s=7}^{48} F_s^* A_{sr} \quad (r = 1, \dots, 6). \quad (197)$$

Kane's Dynamical Equations, then, are  $\tilde{F}_r + \tilde{F}_r^* = 0$ , (198) for  $r = 1, \dots, 6$ .

### CONSTRAINT EQUATIONS

The kinematical and dynamical equations together are fifty-four in number: 48 kinematical, 6 dynamical. Since the complete set of equations for an  $n$ -degree-of-freedom system numbers  $2n$ , and since the system  $\tilde{S}$  has 48 degrees of freedom, 42 more equations are needed to describe completely the motion of the system. These missing equations are the holonomic constraint equations (in nonholonomic form) for the dependent generalized speeds  $u_j^i$  ( $i = 2, \dots, 8; j = 1, \dots, 6$ ).

Since the velocity and angular velocity of the flotor center of mass  $F^*$  is the same irrespective of the actuator path chosen for

describing its position, a set of constraint equations can be written in vector form using the following:

$$s_i \underline{v}^{F^*} = \left( \frac{s_i d \underline{r}^{s,F^*}}{dt} \right) = s_i \underline{v}^{F^*} \quad (i = 1, j = 2, \dots, 8), \quad (199)$$

and  $s_i \underline{\omega}^{F^*} = s_i \underline{\omega}^{F^*} \quad (i = 1, j = 2, \dots, 8). \quad (200)$

If one expands Eqs. (199) and resolves them into a common coordinate system (here, the  $\hat{\underline{f}}_i$  coordinate system), one obtains the following twenty-one (motion) constraint equations:

$$\begin{bmatrix} f_2^1 f_{1j}^1 - f_1^1 f_{2j}^1 \\ -v_3^1 f_{2j}^1 + v_2^1 f_{3j}^1 \\ -v_2^1 f_{1j}^1 + v_1^1 f_{2j}^1 \\ - (v_2^1 - l_3^1) f_{1j}^1 + v_1^1 f_{2j}^1 \\ v_3^1 f_{1j}^1 - v_1^1 f_{3j}^1 \\ -v_3^1 f_{2j}^1 + (v_2^1 + l_3^1) f_{3j}^1 \end{bmatrix}^T \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \\ u_6^1 \end{bmatrix} = \begin{bmatrix} f_2^i f_{1j}^i - f_1^i f_{2j}^i \\ -v_3^i f_{2j}^i + v_2^i f_{3j}^i \\ -v_2^i f_{1j}^i + v_1^i f_{2j}^i \\ - (v_2^i - l_3^i) f_{1j}^i + v_1^i f_{2j}^i \\ v_3^i f_{1j}^i - v_1^i f_{3j}^i \\ -v_3^i f_{2j}^i + (v_2^i + l_3^i) f_{3j}^i \end{bmatrix}^T \begin{bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \\ u_5^i \\ u_6^i \end{bmatrix}, \quad (i = 2, \dots, 8; j = 1, 2, 3). \quad (201)$$

Similarly, if one expands Eqs. (200) and resolves them into a common coordinate system (here again, the  $\hat{\underline{f}}_i$  coordinate system), one obtains the remaining twenty-one (motion) constraint equations:

$$\begin{bmatrix} -f_{3j}^1 \\ f_{1j}^1 \\ f_{3j}^1 \\ f_{3j}^1 \\ f_{2j}^1 \\ f_{1j}^1 \end{bmatrix}^T \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \\ u_6^1 \end{bmatrix} = \begin{bmatrix} -f_{3j}^i \\ f_{1j}^i \\ f_{3j}^i \\ f_{3j}^i \\ f_{2j}^i \\ f_{1j}^i \end{bmatrix}^T \begin{bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ u_4^i \\ u_5^i \\ u_6^i \end{bmatrix}, \quad (i = 2, \dots, 8; j = 1, 2, 3). \quad (202)$$

(It should be noted here that, although Eqs. (201, 202) are in nonholonomic form, the constraints they represent are actually geometric.)

## STATE-SPACE FORM OF THE EQUATIONS OF MOTION

The dynamical, kinematical, and constraint equations can be arranged in the following state-space form:

$$\begin{bmatrix} I & O \\ O & M \end{bmatrix} \begin{Bmatrix} \underline{\dot{q}} \\ \underline{\ddot{u}}^I \end{Bmatrix} = \begin{bmatrix} O & N \\ K & C \end{bmatrix} \begin{Bmatrix} \underline{q} \\ \underline{u}^I \end{Bmatrix} + \begin{bmatrix} O \\ B \end{bmatrix} \{ \underline{i} \} + \begin{bmatrix} O \\ E \end{bmatrix} \{ \underline{d} \}, \quad (203)$$

where the state vector consists of the 48 coordinates  $\underline{q}$  and the 6

independent generalized speeds  $\underline{u}^I$ , associated with actuator #1; the constant submatrices  $M$ ,  $K$ , and  $C$  are system mass, stiffness, and damping matrices, respectively; the symbols  $I$  and  $O$  represent, respectively, an identity matrix and a zero matrix of appropriate dimensions; vector  $\underline{i}$  contains the control currents to the Lorentz coils; and vector  $\underline{d}$  is the disturbance vector. The input matrices  $B$  and  $E$  are time-varying matrix functions of the coordinates.  $N$  is a constant

submatrix that incorporates the kinematical equations and the holonomic constraints.

The disturbance term  $[E] \{ \underline{d} \}$  accounts for the umbilical bias force  $\underline{F}_b$  and moment  $\underline{M}_b$ , and the unknown direct disturbance force  $\underline{F}^D$  and moment  $\underline{M}^D$ . Recall that in the development of the foregoing equations the angular acceleration of the stator was assumed to be negligible. However, the *translational* acceleration of the stator, although presumably unknown, *cannot* be neglected. In fact, that acceleration is the source of the umbilical contribution to flotor g-jitter. To include this indirect disturbance contribution, one simply adds an (unknown) indirect acceleration disturbance term  $\underline{a}^I$  to each of the Eqs. (26), (27), and (28). Along with the other disturbances, this indirect disturbance will appear in the final term of Eq. (203).

## MODEL VALIDATION

AUTOLEV software, marketed by Online Dynamics, Inc., was used to create a full nonlinear model of ARIS, including the actuator (rigid-body) dynamics. AUTOLEV was then used to develop and verify the linearized equations presented in this paper.

The linearized model (without umbilicals) was implemented in MATLAB m-files, using actual values of system parameters. The mathematical model was then checked for kinematical consistency. The procedure used was first to compare the eight position vectors from a common point on the stator to the flotor center of mass, as traced through the eight actuators, with the flotor centered in its home position. The eight position vectors matched exactly. The procedure was repeated with the flotor moved from its home position, in six degrees of freedom. The position vectors tracked within acceptable limits. The linearized MATLAB model was compared with the full nonlinear AUTOLEV model. The *static* response of the linearized model tracked the static response of the nonlinear model, for small angles.

An independent model was developed using the DENEb Envision software, with current CAD models of an ARIS-outfitted ISPR. This model was used as an independent (static) check of the actuator kinematics.

## FUTURE WORK

The next tasks will be the addition of umbilical forces to the AUTOLEV and MATLAB equations, and the *dynamical* validation of the MATLAB model. The procedure will involve simulating the application of various loads to the flotor, and verifying that the eight position vectors track in this dynamic simulation. System dynamics will be incorporated into the Envision model, along with the capability of state-space discrete-time control. The MATLAB and Envision models will then be available, respectively, for centralized, state-space/optimal controller design and for closed-loop system simulation.

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